

## PERFECT DIFFERENCE SETS CONSTRUCTED FROM SIDON SETS

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A set  $\mathcal{A}$  of positive integers is a *perfect difference set* if every nonzero integer has a unique representation as the difference of two elements of  $\mathcal{A}$ . We construct dense *perfect difference sets* from dense Sidon sets. As a consequence of this new approach we prove that there exists a perfect difference set  $\mathcal{A}$  such that

$$A(x) \gg x^{\sqrt{2}-1-o(1)}.$$

Also we prove that there exists a *perfect difference set*  $\mathcal{A}$  such that  $\limsup_{x \rightarrow \infty} A(x)/\sqrt{x} \geq 1/\sqrt{2}$ .

### 1. Introduction

Let  $\mathbb{Z}$  denote the integers and  $\mathbb{N}$  the positive integers. For nonempty sets of integers  $\mathcal{A}$  and  $\mathcal{B}$ , we define the *difference set*

$$\mathcal{A} - \mathcal{B} = \{a - b : a \in \mathcal{A} \text{ and } b \in \mathcal{B}\}.$$

For every integer  $u$ , we denote by  $d_{\mathcal{A}, \mathcal{B}}(u)$  the number of pairs  $(a, b) \in \mathcal{A} \times \mathcal{B}$  such that  $u = a - b$ . Let  $d_{\mathcal{A}}(u)$  the number of pairs  $(a, a') \in \mathcal{A} \times \mathcal{A}$  such that  $u = a - a'$ . The set  $\mathcal{A}$  is a *perfect difference set* if  $d_{\mathcal{A}}(u) = 1$  for every integer  $u \neq 0$ . Note that  $\mathcal{A}$  is a perfect difference set if and only  $d_{\mathcal{A}}(u) = 1$  for every

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positive integer  $u$ . For perfect difference sets, a simple counting argument shows that

$$A(x) \leq (1 + o(1))\sqrt{x},$$

where the *counting function*  $A(x)$  counts the number of positive elements of  $\mathcal{A}$  not exceeding  $x$ .

It is not completely obvious that perfect difference sets exist, but the greedy algorithm produces [4] (see also [7]) a perfect difference set  $\mathcal{A} \subseteq \mathbb{N}$  such that

$$A(x) \gg x^{1/3}.$$

An adaption of the random construction of Sidon sets given in [1] gives the lower bound  $A(x) \gg (x \log x)^{1/3}$  [6]. At the Workshop on Combinatorial and Additive Number Theory (CANT 2004) in New York in May, 2004, Seva Lev (see also [4]) asked if there exists a perfect difference set  $\mathcal{A}$  such that

$$A(x) \gg x^\delta \text{ for some } \delta > 1/3.$$

We answer this question affirmatively by constructing perfect difference sets from classical Sidon sets.

We say that a set  $\mathcal{B}$  is a Sidon set if  $d_{\mathcal{B}}(u) \leq 1$  for all integer  $u \neq 0$ .

**Theorem 1.1.** *For every Sidon set  $\mathcal{B}$  and every function  $\omega(x) \rightarrow \infty$ , there exists a perfect difference set  $\mathcal{A} \subseteq \mathbb{N}$  satisfying*

$$A(x) \geq B(x/3) - \omega(x).$$

It is a difficult problem to construct dense infinite Sidon sets. Ruzsa [9] proved that there exists a Sidon set  $\mathcal{B}$  with  $B(x) \gg x^{\sqrt{2}-1-o(1)}$ . The following result follows easily.

**Theorem 1.2.** *There exists a perfect difference set  $\mathcal{A} \subseteq \mathbb{N}$  such that*

$$A(x) \gg x^{\sqrt{2}-1+o(1)}.$$

Erdős [10] proved that the lower bound  $A(x) \gg x^{1/2}$  does not hold for any Sidon set  $\mathcal{A}$ , and so does not hold for perfect difference sets. However, Krückeberg [3] proved that there exists a Sidon set  $\mathcal{B}$  such that

$$\limsup_{x \rightarrow \infty} \frac{B(x)}{\sqrt{x}} \geq \frac{1}{\sqrt{2}}.$$

We extend this result to perfect difference sets.

**Theorem 1.3.** *There exists a perfect difference set  $\mathcal{A} \subset \mathbb{N}$  such that*

$$\limsup_{x \rightarrow \infty} \frac{A(x)}{\sqrt{x}} \geq \frac{1}{\sqrt{2}}.$$

D. Pollington [6] proved the theorem above with  $1/2$  instead of  $1/\sqrt{2}$ . Notice also that an immediate application of [Theorem 1.1](#) to Krückeberg's result would give only  $\limsup_{x \rightarrow \infty} A(x)x^{-1/2} \geq 1/\sqrt{6}$ .

## 2. Proof of [Theorem 1.1](#)

### 2.1. Sketch of the proof

The strategy of the proof is the following:

- Modify any dense Sidon set  $\mathcal{B}$  given by dilating it by 3 and removing a suitable *thin* subset of  $3*\mathcal{B} = \{3b, b \in \mathcal{B}\}$ .
- Complete the remainder set  $\mathcal{B}_0 = (3*\mathcal{B}) \setminus \{\text{removed set}\}$  with a subset of the elements of a very sparse sequence  $\mathcal{U} = \{u_s\}$  by adding, if  $k$  has not appeared yet in the difference set, two elements  $u_{2k}, u_{2k+1}$  in the  $k$ -th step such that  $u_{2k+1} - u_{2k} = k$ .

### 2.2. The auxiliary sequence $\mathcal{U}$

For any strictly increasing function  $g: \mathbb{N} \rightarrow \mathbb{N}$  and  $k \geq 1$ , we define integers  $u_{2k}$  and  $u_{2k+1}$  by

$$\begin{aligned} u_{2k} &= 4^{g(k)} + \epsilon_k \\ u_{2k+1} &= 4^{g(k)} + \epsilon_k + k \end{aligned}$$

where

$$\epsilon_k = \begin{cases} 1 & \text{if } k \equiv 2 \pmod{3}, \\ 0 & \text{otherwise.} \end{cases}$$

For all positive integers  $k$  we have

$$u_{2k+1} - u_{2k} = k.$$

Let  $\mathcal{U}_k = \{u_{2k}, u_{2k+1}\}$  and  $\mathcal{U}_{<\ell} = \bigcup_{k < \ell} \mathcal{U}_k$ . It will be useful to state some properties of the sequence  $\mathcal{U} = \{u_i\}_{i=2}^{\infty}$ .

**Lemma 2.1.** *The sequence  $\mathcal{U} = \{u_i\}_{i=2}^{\infty}$  satisfies the following properties:*

- (i) For all  $i \geq 2$ ,  $u_i \not\equiv 0 \pmod{3}$ .  
(ii) For all  $k \geq 2$ , for  $u \in \mathcal{U}_k$ , and for all  $u', u'', u''' \in \mathcal{U}_{< k}$ , we have  $u + u' > u'' + u'''$ .  
(iii) If  $k \geq 2$ ,  $u \in \mathcal{U}_k$ , and  $u' \in \mathcal{U}_{< k}$ , then  $u - u' > u/2$ .

**Proof.** (i) By construction.

(ii) Since  $g(k)$  is strictly increasing we have  $k \leq g(k)$  and so

$$4k < 4^k \leq 4^{g(k)}$$

for all  $k \geq 2$ . It follows that

$$\begin{aligned} u'' + u''' &\leq 2u_{2k-1} \leq 2(4^{g(k-1)} + k) \leq 2(4^{g(k)-1} + k) \\ &\leq \frac{4^{g(k)}}{2} + 2k \leq 4^{g(k)} \leq u < u + u'. \end{aligned}$$

(iii) For  $k \geq 2$  we have

$$u' \leq 4^{g(k-1)} + (k-1) + \epsilon_{k-1} \leq 4^{g(k)-1} + k < 2 \cdot 4^{g(k)-1} = \frac{4^{g(k)}}{2} \leq u/2$$

and so  $u - u' > u/2$ . ■

### 2.3. Construction of the Sidon set $\mathcal{B}_0$

Take a Sidon set  $\mathcal{B}$  and consider the set  $\mathcal{B}' = 3*\mathcal{B} = \{3b : b \in \mathcal{B}\}$ . Then  $\mathcal{B}'$  is a Sidon set such that  $b \equiv 0 \pmod{3}$  for all  $b \in \mathcal{B}'$  and  $B'(x) = B\left(\frac{x}{3}\right)$ .

The set  $\mathcal{B}_0$  will be the set  $\mathcal{B}' = 3*\mathcal{B}$  after we remove all the elements  $b \in \mathcal{B}'$  that satisfy at least one of the followings conditions:

- (c1)  $b = u - u' + b'$  for some  $b' \in \mathcal{B}'$ ,  $b > b'$  and  $u, u' \in \mathcal{U}$  such that  $u \in \mathcal{U}_r$ ,  $u' \in \mathcal{U}_{< r}$  for some  $r$ .  
(c2)  $b = u + u' - b'$  for some  $b' \in \mathcal{B}'$ ,  $b \geq b'$  and  $u, u' \in \mathcal{U}$ .  
(c3)  $b = u + u' - u''$  for some  $u \in \mathcal{U}_r$ ,  $u' \in \mathcal{U}$ , and  $u'' \in \mathcal{U}_{< r}$  with  $u' \leq u$ .  
(c4)  $|b - u_i| \leq i$  for some  $u_i \in \mathcal{U}$ .

### 2.4. The inductive step

We shall construct the set  $\mathcal{A}$  in [Theorem 1.1](#) by adjoining terms to the *nice* Sidon set  $\mathcal{B}_0$  obtained above. More precisely, the sequence  $\mathcal{A}$  satisfying the conditions of the theorem will be

$$\mathcal{A} = \bigcup_{k=0}^{\infty} \mathcal{A}_k$$

where  $\mathcal{A}_k$  will be defined by  $\mathcal{A}_0 = \mathcal{B}_0$  and for  $k \geq 1$ ,

$$\mathcal{A}_k = \begin{cases} \mathcal{A}_{k-1} \cup \mathcal{U}_k & \text{if } k \notin \mathcal{A}_{k-1} - \mathcal{A}_{k-1} \\ \mathcal{A}_{k-1} & \text{otherwise.} \end{cases}$$

**Lemma 2.2.** *For every positive integer  $k$  we have*

$$[-k, k] \subseteq \mathcal{A}_k - \mathcal{A}_k$$

and so

$$d_{\mathcal{A}}(n) \geq 1$$

for all integers  $n$ .

**Proof.** Clear. ■

## 2.5. $\mathcal{A}$ is a Sidon set

First we state two lemmas.

**Lemma 2.3.** *Let  $A_1$  and  $A_2$  be nonempty disjoint sets of integers and let  $A = A_1 \cup A_2$ . For every integer  $n$  we have*

$$d_A(n) = d_{A_1}(n) + d_{A_2}(n) + d_{A_1, A_2}(n) + d_{A_2, A_1}(n),$$

where

$$d_{A_i, A_j}(n) = \#\{(a, a') \in A_i \times A_j, a - a' = n\}.$$

**Proof.** This follows from the identity

$$(A_1 \cup A_2) \times (A_1 \cup A_2) = (A_1 \times A_1) \cup (A_2 \times A_2) \cup (A_1 \times A_2) \cup (A_2 \times A_1). \quad \blacksquare$$

**Lemma 2.4.** *If  $n \in \mathcal{A}_{r-1} - \mathcal{U}_r$  then*

- (i)  $|n| > r$ , and so  $d_{\mathcal{U}_r, \mathcal{A}_{r-1}}(r) = d_{\mathcal{A}_{r-1}, \mathcal{U}_r}(r) = 0$ .
- (ii)  $d_{\mathcal{A}_{r-1}}(n) = 0$ .
- (iii)  $d_{\mathcal{U}_r, \mathcal{A}_{r-1}}(n) = 0$ .

**Proof.** Write  $n = a - u$ , where  $a \in \mathcal{A}_{r-1}$  and  $u \in \mathcal{U}_r = \{u_{2r}, u_{2r+1}\}$ .

(i) If  $a = b \in \mathcal{B}_0$  we have that  $|b - u| > 2r > r$  because, by condition (c4), we have removed all elements  $b$  from  $\mathcal{B}$  such that  $|b - u_i| \leq i$ .

If  $a = u' \in \mathcal{U}_{<r}$  then we apply Lemma 2.1 (iii) to conclude that

$$|u' - u| > \frac{u}{2} \geq \frac{4^{g(r)}}{2} > r.$$

(ii) Since  $\mathcal{A}_{r-1} \subseteq \mathcal{B}_0 \cup \mathcal{U}_{<r}$ , it follows that

$$d_{\mathcal{A}_{r-1}}(n) \leq d_{\mathcal{B}_0 \cup \mathcal{U}_{<r}}(n) \leq d_{\mathcal{B}_0}(n) + d_{\mathcal{U}_{<r}}(n) + d_{\mathcal{B}_0, \mathcal{U}_{<r}}(n) + d_{\mathcal{U}_{<r}, \mathcal{B}_0}(n).$$

If  $a = b \in \mathcal{B}_0$ , then  $n = b - u$  and

1.  $b \equiv 0 \pmod{3}$  but  $u \not\equiv 0 \pmod{3}$ , hence  $b - u \not\equiv 0 \pmod{3}$  and  $d_{\mathcal{B}_0}(b - u) = 0$  (by Lemma 2.1 (i));
2.  $d_{\mathcal{U}_{<r}}(b - u) = 0$  (by condition (c3));
3.  $d_{\mathcal{B}_0, \mathcal{U}_{<r}}(b - u) = 0$  (by condition (c1));
4.  $d_{\mathcal{U}_{<r}, \mathcal{B}_0}(b - u) = 0$  (by condition (c2)).

If  $a = u' \in \mathcal{U}_{<r}$ , then  $n = u' - u$  and

1.  $d_{\mathcal{B}_0}(u' - u) = 0$  (by condition (c1));
2.  $d_{\mathcal{U}_{<r}}(u' - u) = 0$  (by Lemma 2.1 (ii));
3. if  $u' - u = b - u''$  with  $u'' \in \mathcal{U}_{<r}$ , then Lemma 2.1 (iii) implies that  $0 < b = u' + u'' - u \leq 0$ , and so  $d_{\mathcal{B}_0, \mathcal{U}_{<r}}(u' - u) = 0$ ;
4.  $d_{\mathcal{U}_{<r}, \mathcal{B}_0}(u' - u) = 0$  (by condition (c3)).

(iii) Again, since  $\mathcal{A}_{r-1} \subseteq \mathcal{B}_0 \cup \mathcal{U}_{<r}$  we have that

$$d_{\mathcal{U}_r, \mathcal{A}_{r-1}}(n) \leq d_{\mathcal{U}_r, \mathcal{B}_0}(n) + d_{\mathcal{U}_r, \mathcal{U}_{<r}}(n).$$

If  $a = b \in \mathcal{B}_0$  then  $d_{\mathcal{U}_r, \mathcal{B}_0}(b - u) = 0$  (by condition (c2)) and  $d_{\mathcal{U}_r, \mathcal{U}_{<r}}(b - u) = 0$  (by condition (c3)).

If  $a = u' \in \mathcal{U}_{<r}$  then  $d_{\mathcal{U}_r, \mathcal{B}_0}(u' - u) = 0$  (by condition (c3)). Finally, we have  $d_{\mathcal{U}_r, \mathcal{U}_{<r}}(u' - u) = 0$ , since if  $u' - u = u'' - u'''$ ,  $u'' \in \mathcal{U}_r$ ,  $u''' \in \mathcal{U}_{<r}$ , then  $0 > u' - u = u'' - u''' > 0$ . This completes the proof. ■

**Lemma 2.5.** *For every positive integer  $n$  we have*

$$d_{\mathcal{A}}(n) \leq 1$$

and so  $\mathcal{A}$  is a perfect difference set.

**Proof.** We will use induction to prove that, for every  $r \geq 0$ ,

$$d_{\mathcal{A}_r}(n) \leq 1 \quad \text{for every nonzero integer } n.$$

This is true for  $r = 0$  because  $\mathcal{A}_0 = \mathcal{B}_0$  is a subset of a Sidon set.

We assume that the statement is true for  $r - 1$  and shall prove it for  $r$ .

If  $d_{\mathcal{A}_{r-1}}(r) = 1$  then  $\mathcal{A}_r = \mathcal{A}_{r-1}$  and there is nothing to prove. Suppose that  $d_{\mathcal{A}_{r-1}}(r) = 0$ , and so  $\mathcal{A}_r = \mathcal{A}_{r-1} \cup \mathcal{U}_r$ . Since we have added two new elements  $u_{2r}, u_{2r+1}$  to  $\mathcal{A}_{r-1}$ , it is possible that there are *new* representations of a positive integer  $n$  so that  $d_{\mathcal{A}_r}(n) > 1$ . We shall prove that this cannot happen.

By Lemma 2.3, we can write

$$d_{\mathcal{A}_r}(n) = d_{\mathcal{A}_{r-1}}(n) + d_{\mathcal{U}_r}(n) + d_{\mathcal{A}_{r-1}, \mathcal{U}_r}(n) + d_{\mathcal{U}_r, \mathcal{A}_{r-1}}(n).$$

If  $n=r$ , then Lemma 2.4 (i) and the relation  $u_{2r+1} - u_{2r} = r$  imply that

$$d_{\mathcal{A}_r}(r) = d_{\mathcal{A}_{r-1}}(r) + d_{\mathcal{U}_r}(r) + d_{\mathcal{A}_{r-1}, \mathcal{U}_r}(r) + d_{\mathcal{U}_r, \mathcal{A}_{r-1}}(r) = 0 + 1 + 0 + 0 = 1.$$

If  $n \neq r$ , then

$$d_{\mathcal{A}_r}(n) = d_{\mathcal{A}_{r-1}}(n) + d_{\mathcal{A}_{r-1}, \mathcal{U}_r}(n) + d_{\mathcal{U}_r, \mathcal{A}_{r-1}}(n).$$

If  $n \in \mathcal{A}_{r-1} - \mathcal{U}_r$  (the case  $n \in \mathcal{U}_r - \mathcal{A}_{r-1}$  is similar), then we can write

$$n = a - u \text{ where } a \in \mathcal{A}_{r-1}, u \in \mathcal{U}_r.$$

Applying Lemma 2.4 (ii) and Lemma 2.4 (iii), we obtain

$$d_{\mathcal{A}_r}(n) = d_{\mathcal{A}_{r-1}, \mathcal{U}_r}(n).$$

If  $d_{\mathcal{A}_{r-1}, \mathcal{U}_r}(n) \geq 2$ , then there exist  $a, a' \in \mathcal{A}_{r-1}$  such that  $a - u_{2r} = a' - u_{2r+1}$ . This implies that

$$a' - a = u_{2r+1} - u_{2r} = r \in \mathcal{A}_{r-1} - \mathcal{A}_{r-1}$$

which is false, so  $d_{\mathcal{A}_r}(n) = d_{\mathcal{A}_{r-1}, \mathcal{U}_r}(n) \leq 1$ .

If  $n \notin (\mathcal{A}_{r-1} - \mathcal{U}_r) \cup (\mathcal{U}_r - \mathcal{A}_{r-1})$  then

$$d_{\mathcal{A}_r}(n) = d_{\mathcal{A}_{r-1}}(n) \leq 1.$$

This completes the proof. ■

## 2.6. The counting function $A(x)$

We have

$$A(x) \geq B_0(x) = B'(x) - R(x) = B(x/3) - R(x)$$

where  $R = R_1 \cup R_2 \cup R_3 \cup R_4$  and  $R_i$  denotes the set of elements of  $B$  removed by condition (ci),  $i = 1, 2, 3, 4$ .

**Lemma 2.6.** *Let  $U(x)$  denote the counting function of the set  $\mathcal{U}$ . For the sets  $R_1, R_2, R_3, R_4$  defined above, we have*

- (i)  $R_1(x) \leq U^2(2x)$ ,
- (ii)  $R_2(x) \leq U^2(2x)$ ,
- (iii)  $R_3(x) \leq U^3(2x)$ ,
- (iv)  $R_4(x) \leq 2U^2(2x) + U(2x)$ .

**Proof.** (i) We have

$$R_1(x) = \#\{b \in \mathcal{B}' : b \leq x \text{ and } b \text{ satisfies condition (c1)}\}.$$

Because  $\mathcal{B}'$  is a Sidon set, for every pair of integers  $u, u' \in \mathcal{U}$  there exists at most one pair of integers  $b, b' \in \mathcal{B}'$  such that  $b - b' = u - u'$ . The condition  $x \geq b > b'$  implies that  $0 < u - u' \leq x$ . On the other hand Lemma 2.1 (iii) implies that  $u - u' > u/2$  and so  $u < 2x$  and

$$R_1(x) \leq \#\{(u, u'), u' < u, u < 2x\} \leq U^2(2x).$$

(ii) Again, because  $\mathcal{B}'$  is a Sidon set, for every pair  $u, u' \in \mathcal{U}$  there exists at most one pair  $b, b' \in \mathcal{B}'$  such that  $b + b' = u + u'$ . The condition  $x \geq b \geq b'$  implies  $u, u' \leq 2x$  and so

$$R_2(x) \leq \#\{(u, u') \in \mathcal{U} \times \mathcal{U} : u \leq 2x, u' \leq 2x\} \leq U^2(2x).$$

(iii) If  $u \in \mathcal{U}_r$ ,  $u'' \in \mathcal{U}_{<r}$ , then Lemma 2.1 (iii) implies that  $b = u + u' - u'' > u - u'' > u/2$  and so

$$\begin{aligned} R_3(x) &= \#\{b \in \mathcal{B}' : b \leq x \text{ and } b \text{ satisfies condition (c3)}\} \\ &\leq \#\{(u, u', u'') \in \mathcal{U} \times \mathcal{U} \times \mathcal{U} : u < 2x, u'' < u, u' \leq u\} \\ &\leq U(2x)^3. \end{aligned}$$

(iv) We have

$$\begin{aligned} R_4(x) &= \#\{b \in \mathcal{B}' : b \leq x \text{ and } |b - u_i| \leq i \text{ for some } u_i \in \mathcal{U}\} \\ &\leq \#\{n \in \mathbb{N} : n \leq x \text{ and } |n - u_i| \leq i \text{ for some } i\}. \end{aligned}$$

If  $n \leq x$  and  $|n - u_i| \leq i$ , then  $u_i \leq n + i \leq x + i$ . Since  $u_2 = 4^{g(1)} \geq 4$ ,  $u_3 = 4^{g(1)+1} \geq 16$ , and, for  $i \geq 4$ ,

$$u_i \geq 4^{g((i-1)/2)} \geq 4^{(i-1)/2} = 2^{i-1} \geq 2i.$$

Therefore,  $u_i \leq x + i \leq x + u_i/2$  and so  $u_i \leq 2x$ . It follows that  $i \leq U(2x)$  and so

$$\begin{aligned} R_4(x) &\leq \#\{n \leq x : |n - u_i| \leq U(2x) \text{ and } u_i \leq 2x\} \leq (2U(2x) + 1)U(2x) \\ &= 2U(2x)^2 + U(2x). \end{aligned}$$

This completes the proof of the lemma. ■

Finally, given any function  $\omega(x) \rightarrow \infty$  we have that

$$A(x) \geq B(x/3) - (U(2x)^3 + 4U^2(2x) + U(2x)) \geq B(x/3) - \omega(x)$$

for any function  $g: \mathbb{N} \rightarrow \mathbb{N}$  and sequence  $\mathcal{U}$  growing fast enough. This completes the proof of Theorem 1.1.

### 3. Proof of Theorem 1.3

**Lemma 3.1.** *If  $C_1$  and  $C_2$  are Sidon sets such that  $(C_i - C_i) \cap (C_j - C_j) = \{0\}$ ,  $(C_i + C_i) \cap (C_j + C_j) = \emptyset$  and  $(C_i + C_i - C_i) \cap C_j = \emptyset$  for  $i \neq j$ , then  $C_1 \cup C_2$  is a Sidon set.*

**Proof.** Obvious. ■

**Lemma 3.2.** *For each odd prime  $p$  there exist a Sidon set  $\mathcal{B}_p$  such that*

- (i)  $\mathcal{B}_p \subseteq [1, p^2 - p]$ ,
- (ii)  $(\mathcal{B}_p - \mathcal{B}_p) \cap [-\sqrt{p}, \sqrt{p}] = \emptyset$ ,
- (iii)  $|\mathcal{B}_p| > p - 2\sqrt{p}$ .

**Proof.** Ruzsa [8] constructed, for each prime  $p$ , a Sidon set  $R_p \subseteq [1, p^2 - p]$  with  $|R_p| = p - 1$ . We consider the subset  $\mathcal{B}_p$  of  $R_p$  that we obtain by removing all elements  $b \in R_p$  such that  $0 < |b - b'| \leq \sqrt{p}$  for some  $b' \in R_p$ . Since  $R_p$  is a Sidon set, it follows that we have removed at most  $\sqrt{p}$  elements from  $R_p$ , and so  $|\mathcal{B}_p| \geq |R_p| - \sqrt{p} = p - \sqrt{p} - 1 > p - 2\sqrt{p}$ . ■

**Proof of Theorem 1.3.** We shall construct an increasing sequence of finite set  $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$  such that  $\mathcal{A} = \bigcup_{k=1}^{\infty} A_k$  is a perfect difference set satisfying Theorem 1.3.

In the following,  $l_k$  will denote the largest integer in the set  $A_{k-1}$ , and  $p_k$  the least prime greater than  $4l_k^2$ . Thus  $l_k < \sqrt{p_k}/2$ . Let

$$A_1 = \{0, 1\}.$$

Then  $l_2 = 1$  and  $p_2 = 5$ . We define

$$A_k = \begin{cases} A_{k-1} \cup (\mathcal{B}_{p_k} + p_k^2 + 2l_k) & \text{if } k \in A_{k-1} - A_{k-1} \\ A_{k-1} \cup (\mathcal{B}_{p_k} + p_k^2 + 2l_k) \cup \{4p_k^2, 4p_k^2 + k\} & \text{otherwise,} \end{cases}$$

with  $\mathcal{B}_{p_k}$  defined as in Lemma 3.2. We shall prove that the set  $\mathcal{A} = \bigcup_{k=1}^{\infty} A_k$  satisfies the theorem.

By construction,  $[1, k] \subseteq A_k - A_k$  for every positive integer  $k$  and so  $\mathcal{A} - \mathcal{A} = \mathbb{Z}$ .

We must prove that  $A_k$  is a Sidon set for every  $k \geq 1$ .

This is clear for  $k = 1$ . Suppose that  $A_{k-1}$  is a Sidon set. Let  $C_1 = A_{k-1}$  and  $C_2 = \mathcal{B}_{p_k} + p_k^2 + 2l_k$ . We shall show that

$$C_1 \cup C_2 = A_{k-1} \cup (\mathcal{B}_{p_k} + p_k^2 + 2l_k)$$

is a Sidon set applying [Lemma 3.1](#). Notice that

$$C_1 - C_1 \subseteq [-l_k, l_k] \subseteq [-\sqrt{p_k}, \sqrt{p_k}]$$

$$C_2 - C_2 = \mathcal{B}_{p_k} - \mathcal{B}_{p_k}$$

$$[-\sqrt{p_k}, \sqrt{p_k}] \cap (\mathcal{B}_{p_k} - \mathcal{B}_{p_k}) = \{0\}.$$

Then

$$(C_1 - C_1) \cap (C_2 - C_2) = \{0\}.$$

Notice also that if  $x \in C_2 + C_2$  then  $x \geq 2p_k^2 + 4l_k$ , but  $C_1 + C_1 \subset [1, 2l_k]$ . Then

$$(C_1 + C_1) \cap (C_2 + C_2) = \emptyset.$$

If  $x \in (C_1 + C_1 - C_1)$ , then  $x \leq 2l_k$ , but if  $x \in C_2$ , then  $x > 2l_k$ . Thus,

$$(C_1 + C_1 - C_1) \cap C_2 = \emptyset.$$

If  $x \in C_2 + C_2 - C_2$ , then  $x \geq 2(p_k^2 + 2l_k + 1) - (p_k^2 + p_k^2 + 2l_k) = 2l_k + 2$ , and if  $x \in C_1$ , then  $x \leq l_k$ . Therefore,

$$(C_2 + C_2 - C_2) \cap C_1 = \emptyset.$$

Then  $A_{k-1} \cup (\mathcal{B}_{p_k} + p_k^2 + 2l_k)$  is a Sidon set.

Now we must distinguish two cases:

If  $k \in A_{k-1} - A_{k-1}$  then  $A_k = A_{k-1} \cup (\mathcal{B}_{p_k} + p_k^2 + 2l_k)$  and we have proved that it is a Sidon set.

If  $k \notin A_{k-1} - A_{k-1}$  then  $A_k = A_{k-1} \cup (\mathcal{B}_{p_k} + p_k^2 + 2l_k) \cup \{4p_k^2, 4p_k^2 + k\}$  and we have to prove that it is also a Sidon set. In this case we take  $C_1 = A_{k-1} \cup (\mathcal{B}_{p_k} + p_k^2 + 2l_k)$  and  $C_2 = \{4p_k^2, 4p_k^2 + k\}$ . We can write

$$\begin{aligned} C_1 - C_1 &= (A_{k-1} - A_{k-1}) \cup (\mathcal{B}_{p_k} - \mathcal{B}_{p_k}) \cup (A_{k-1} - (\mathcal{B}_{p_k} + p_k^2 + 2l_k)) \\ &\quad \cup ((\mathcal{B}_{p_k} + p_k^2 + 2l_k) - A_{k-1}). \end{aligned}$$

If  $x \in (A_{k-1} - (\mathcal{B}_{p_k} + p_k^2 + 2l_k)) \cup ((\mathcal{B}_{p_k} + p_k^2 + 2l_k) - A_{k-1})$ , then  $|x| \geq p_k^2 + l_k > k$ .

If  $x \in (\mathcal{B}_{p_k} - \mathcal{B}_{p_k})$  then  $x = 0$  or  $|x| > \sqrt{p_k} > 2l_k > k$ , then, since  $C_2 - C_2 = \{-k, 0, k\}$ , we have

$$(C_1 - C_1) \cap (C_2 - C_2) = \{0\}.$$

On the other hand if  $x \in C_2 + C_2$  then  $x \geq 8p_k^2$  but

$$C_1 + C_1 \subset [1, 2(2p_k^2 - p_k + 2l_k)] \subset [1, 4p_k^2].$$

Then

$$(C_1 + C_1) \cap (C_2 + C_2) = \emptyset.$$

If  $x \in C_1 + C_1 - C_1$  then  $x \leq 2(2p_k^2 - p_k + 2l_k) < 4p_k^2$ . Thus,

$$(C_1 + C_1 - C_1) \cap C_2 = \emptyset.$$

Also we have that  $C_2 + C_2 - C_2 = 4p_k^2 + \{-k, 0, k, 2k\}$ , but if  $x \in C_1$  we have that  $x < 2p_k^2 - p_k + 2l_k < 2p_k^2 - 4l_k^2 + 2l_k < 4p_k^2 - 2l_k^2 < 4p_k^2 - k$ . Thus

$$(C_2 + C_2 - C_2) \cap C_1 = \emptyset.$$

To finish the proof of the theorem note that

$$\begin{aligned} \limsup_{x \rightarrow \infty} \frac{\mathcal{A}(x)}{\sqrt{x}} &\geq \limsup_{k \rightarrow \infty} \frac{\mathcal{A}(2p_k^2 - p_k + l_k)}{\sqrt{2p_k^2 - p_k + l_k}} \geq \limsup_{k \rightarrow \infty} \frac{|\mathcal{B}_{p_k}|}{\sqrt{2p_k^2 - p_k + l_k}} \\ &\geq \limsup_{k \rightarrow \infty} \frac{p_k - 2\sqrt{p_k}}{\sqrt{2p_k^2 - p_k + \sqrt{p_k}/2}} = \frac{1}{\sqrt{2}}. \end{aligned} \quad \blacksquare$$

## 4. Remarks and Open problems

### 4.1. The sequence $t(\mathcal{A})$ associated to a perfect difference set

Any translation of a perfect difference set intersects to itself in exactly one element, and so we can define, for every perfect difference set  $\mathcal{A}$ , a sequence  $t(\mathcal{A})$  whose elements are given by  $t_n = \mathcal{A} \cap (\mathcal{A} - n)$  for all  $n \geq 1$ . The sequence  $t_n$  is very irregular, but the greedy algorithm used in [4] generates a perfect difference set such that  $t_n \ll n^3$ . Our method generates a dense Sidon set  $\mathcal{A}$ , but gives a very poor upper bound for the sequence  $t_n$ .

**Problem 4.1.** Does there exists perfect difference set such that  $t_n = o(n^3)$ ?

### 4.2. Sidon sets included in perfect difference sets

We have proved that any Sidon set can be perturbed slightly to become a subset of a perfect difference set. Every subset of a perfect difference set is a Sidon set. It is natural to ask if *every* Sidon set is a subset of a perfect difference set. The answer is negative. To construct a counterexample, we take a perfect difference set  $\mathcal{A}$  and consider the set  $\mathcal{B} = 2 * \mathcal{A} = \{2a : a \in \mathcal{A}\}$ . The set  $\mathcal{B}$  has the following properties:

- (i)  $\mathcal{B}$  is a Sidon set.
- (ii) If  $n$  is an even integer not in  $\mathcal{B}$ , then  $\mathcal{B} \cup \{n\}$  is not a Sidon set.
- (iii) If  $m$  and  $m'$  are distinct odd integers not in  $\mathcal{B}$ , then  $\mathcal{B} \cup \{m, m'\}$  is not a Sidon set.

The Sidon set  $\mathcal{B}$  is not a subset of a perfect difference set. Since this construction is rather artificial, we wonder if almost all Sidon sets are subsets of perfect difference sets.

**Problem 4.2.** Determine when a Sidon set is a subset of a perfect difference set.

### 4.3. Perfect $h$ -sumsets

Let  $\mathcal{A}$  be a set of integers. For every integer  $u$ , we denote by  $r_{\mathcal{A}}^h(u)$  the number of  $h$ -tuples  $(a_1, \dots, a_h) \in \mathcal{A}^h$ , such that

$$a_1 \leq \dots \leq a_h$$

and

$$a_1 + \dots + a_h = u.$$

We say that  $\mathcal{A}$  is a *perfect  $h$ -sumset* or a *unique representation basis of order  $h$*  if  $r_{\mathcal{A}}^h(u) = 1$  for every integer  $u$ . Nathanson [5] proved that for every  $h \geq 2$  and for every function  $f: \mathbb{Z} \rightarrow \mathbb{N}_0 \cup \{\infty\}$  such that  $\limsup_{|u| \rightarrow \infty} f(u) \geq 1$  there exists a set of integers  $\mathcal{A}$  such that

$$r_{\mathcal{A}}^h(u) = f(u)$$

for every integer  $u$ . In particular, the *perfect  $h$ -sumsets* correspond to the representation function  $f \equiv 1$ . Nathanson's construction produces a *perfect  $h$ -sumset*  $\mathcal{A}$  with

$$A(x) \gg x^{1/(2h-1)}$$

and he asked for denser constructions.

It is easy to modify our approach to get a perfect 2-sumset  $\mathcal{A}$  with  $A(x) \gg x^{\sqrt{2}-1+o(1)}$ . But for  $h \geq 3$  our method cannot be adapted easily, and a more complicated construction is needed. We shall study perfect  $h$ -sumsets in a forthcoming paper [2].

#### 4.4. Sums and differences

Let  $\mathcal{A}$  be a set of integers. For every integer  $u$ , we denote by  $d_{\mathcal{A}}(u)$  and  $s_{\mathcal{A}}(u)$  the number of solutions of

$$u = a - a' \quad \text{with } a, a' \in \mathcal{A}$$

and

$$u = a + a' \quad \text{with } a, a' \in \mathcal{A} \text{ and } a \leq a',$$

respectively. We say that  $\mathcal{A}$  is a *perfect difference sumset* if  $d_{\mathcal{A}}(n) = 1$  for all  $n \in \mathbb{N}$  and if  $s_{\mathcal{A}}(n) = 1$  for all  $n \in \mathbb{Z}$ .

We can extend [Theorem 1.1](#) and [Theorem 1.3](#) to perfect difference sumsets. Then it is a natural to ask if, for any two functions  $f_1 : \mathbb{N} \rightarrow \mathbb{N}$  and  $f_2 : \mathbb{Z} \rightarrow \mathbb{N}$ , there exists a set  $\mathcal{A}$  such that  $d_{\mathcal{A}}(n) = f_1(n)$  for all  $n \in \mathbb{N}$  and  $s_{\mathcal{A}}(n) = f_2(n)$  for all  $n \in \mathbb{Z}$ . (Note that perfect difference sumsets correspond to the functions  $f_1 \equiv 1$  and  $f_2 \equiv 1$ .) It is not difficult to guess that the answer is no. For example, if  $s_{\mathcal{A}}(n) = 2$  for infinitely many integers  $n$ , it is easy to see that  $d_{\mathcal{A}}(n) \geq 2$  for infinitely many integers  $n$ .

**Problem 4.3.** Give general conditions for functions  $f_1$  and  $f_2$  to assure that there exists a set  $\mathcal{A}$  such that  $d_{\mathcal{A}}(n) \equiv f_1(n)$  and  $s_{\mathcal{A}}(n) \equiv f_2(n)$ .

Is the condition  $\liminf_{u \rightarrow \infty} f_1(u) \geq 2$  and  $\liminf_{|u| \rightarrow \infty} f_2(u) \geq 2$  sufficient?

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